which makes the problem of interest. With this supposition, the pair of equations may be written as

$$xy = a,$$
$$\left(\frac{x+y}{x-y}\right)^2 + \left(\frac{x-y}{x+y}\right)^2 = 2b,$$

where $a = 2$, $b = 41/9$. It is then to be noted that if (r, s) be one solution, the following are the eight distinct solutions:

(r, s), (s, r), $(-r, -s)$, $(-s, -r)$, $(ir, -is)$, $(is, -ir)$, $(-ir, is)$, $(-is, ir)$.

Thus there is a group of eight operations carrying any solution into a solution. This remark applies for all finite non-vanishing values of a, b. Solutions can coincide in pairs only for a, or b, zero or infinite, or for $b = -1$. Since $(1, 2)$ is by inspection a solution, the eight solutions may be written out at once.

By direct methods also the solution is readily obtained. Put $u = (x + y)/(x - y)$; whence

$$x^2 = a(u + 1)/(u - 1),$$
$$y^2 = a(u - 1)/(u + 1),$$

where the signs of x and y are to be so taken that $xy = a$. Then

$$u^2 + 1/u^2 = 2b, \qquad u = \frac{\pm\sqrt{b+1} \pm \sqrt{b-1}}{\sqrt{2}}$$

In the given example $u = 3, -3, \frac{1}{3}$, or $-\frac{1}{3}$, $x^2 = 4, 1, -4, -1$, $y^2 = 1, 4, -1, -4$, and the solutions are

$$(2, 1), \quad (-2, -1), \quad (1, 2), \quad (-1, -2), \quad (2i, -i), \quad (-2i, i), \quad (i, -2i), \quad (-i, 2i).$$

NOTE.—Miss Berta M. King remarks that if the sign in the second equation is changed as stated in Professor Bennett's solution, the problem is in Hawkes's *Advanced Algebra.* She gets the same results as Professor Bennett.

H. N. CARLETON, A. R. NAUER, J. W. SHUMAN and F. L. WILMER solved the problem as it is printed above.

2898 [1921, 228]. Proposed by J. W. CLAWSON, Ursinus College.

Four straight lines determine four triangles. It is well known that the circumcenters of these triangles lie on a circle and that the circumcircles intersect this circle in a point, called the Wallace point. It is also well known that the orthocenters of the four triangles lie in a straight line, which is perpendicular to the line on which lie the middle points of the three diagonals of the quadrilateral determined by the four given straight lines.

Prove that the centroids of the four triangles lie on a parabola whose axis is parallel to the mid-diagonal line; and that the distance from the Wallace point to the mid-diagonal line is two thirds of the distance from the Wallace point to the axis of the parabola.

SOLUTION BY THE PROPOSER.

In the statement of the problem "two thirds" should be "three halves."

Given the four lines, let the origin be at the Wallace point. The Wallace or pedal lines of this point with respect to the four triangles are coincident. Let the y-axis be parallel to this line, its equation being, say, $x = p$. Then the perpendicular from the origin on the four given lines will meet them in four points on this line, which we may call (p, a), (p, b), (p, c), (p, d), and the equations of the four given lines will be

$$px + ay = p^2 + a^2, \text{ etc.}$$

The intersections of these lines are the six points

$$p - \frac{ab}{p},\ a + b, \text{ etc.},$$

and the mid-points of the three diagonals are

$$p - \frac{ab + cd}{2p},\ \tfrac{1}{2}(a + b + c + d), \text{ etc.},$$

so that the equation of the mid-diagonal line is $y = \frac{1}{2}(a + b + c + d)$. Also the centroids of the four triangles are

$$p - \frac{ab + ac + bc}{3p},\ \tfrac{2}{3}(a + b + c), \text{ etc.}$$

The equation of a parabola whose axis is praallel to the x-axis is of the form[1]

$$y^2 + Dx + Ey + F = 0;$$

and if we substitute the coördinates of any three of the centroids, evaluate D, E, F, and simplify, we obtain the equation

$$\left(y - \frac{a + b + c + d}{3}\right)^2 = -\frac{4p}{3}\left(x - p - \frac{a^2 + b^2 + c^2 + d^2}{12p} + \frac{ab + ac + ad + bc + bd + cd}{6p}\right),$$

symmetrical with respect to a, b, c, d.

The axis of this parabola is $y = \frac{1}{3}(a + b + c + d)$; therefore, the distance of the mid-diagonal line from O is three halves of the distance of this axis from O.

We may note that the four orthocenters are

$$2p,\ a + b + c + \frac{abc}{p^2},\ \text{etc.},$$

all lying on the line $x = 2p$, which is twice as far from o as the pedal line.

2791 [1919, 414; 1921, 143–145].

A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at the same rate. When the wine cup is empty, what part of the contents of the lower cup is water? [Proposed by Charles Gilpin, Jr., Philadelphia, as Problem 287 in *The Mathematical Visitor*, January, 1881, volume 1, page 193. No solution was published in the *Visitor*.]

Remarks by R. E. Moritz, University of Washington.

Each of the solutions of the problem published in this Monthly (*1921*) proceeds on the assumption that the liquids flow from the cups, in other words, that the process by which the wine cup is emptied is a continuous process. This assumption seems to me unwarranted, being a flat contradiction of one of the conditions of the problem. The problem states explicitly that "the wine drips into the water, and the mixture drips out at the same rate." Dripping is essentially a discontinuous process, the solutions referred to must therefore be considered approximations rather than exact solutions.

Curiously enough, the exact solution is very much simpler than the approximations in question. Suppose that each cup contains at the outset n drops of equal size. After the first drop has fallen into the lower cup, the mixture consists of n drops of water and one drop of wine,[2] $n/(n + 1)$ of the contents of the lower cup is therefore water; after the second drop of wine has been added, the water content is $[n/(n + 1)]^2$; after the third drop has been added, $[n/(n + 1)]^3$; after the nth drop has been added, that is when the cup has been emptied, $[n/(n + 1)]^n$. It is clear then that the exact answer to the problem is a function of n, the number of drops in the wine cup at the outset.

The limit of $[n/(n + 1)]^n$ as n approaches ∞ is e^{-1}, which is the result obtained in the previous solutions. But n cannot be infinite, for if it were, the wine cup could not be emptied by a dripping process.

[1] That the coördinates of the four centroids satisfy an equation of this type may be seen from the fact that if we substitute them all and eliminate D, E and F, we get a polynomial of degree 5 in a, b, c, d, divisible by the six differences, $a - b$, etc., and, therefore, zero—Editors.

[2] This assumes that the cup can hold $n + 1$ drops till they are mixed—Editors.